

## A NEW BASIS FOR THE SPACE OF MODULAR FORMS

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ABSTRACT. Let  $G_{2n}$  be the Eisenstein series of weight  $2n$  for the full modular group  $\Gamma = SL_2(\mathbb{Z})$ . It is well-known that the space  $M_{2k}$  of modular forms of weight  $2k$  on  $\Gamma$  has a basis  $\{G_4^\alpha G_6^\beta \mid \alpha, \beta \in \mathbb{Z}, \alpha, \beta \geq 0, 4\alpha + 6\beta = 2k\}$ . In this paper we will exhibit another (simpler) basis for  $M_{2k}$ . It is given by  $\{G_{2k}\} \cup \{G_{4i}G_{2k-4i} \mid i = 1, 2, \dots, d_k\}$  if  $2k \equiv 0 \pmod{4}$ , and  $\{G_{2k}\} \cup \{G_{4i+2}G_{2k-4i-2} \mid i = 1, 2, \dots, d_k\}$  if  $2k \equiv 2 \pmod{4}$  where  $d_k + 1 = \dim_{\mathbb{C}} M_{2k}$ .

## 1. INTRODUCTION AND STATEMENT OF RESULTS

Modular forms of one variable have been studied for a long time. They appear in many areas of mathematics and in theoretical physics. In this paper we consider the space  $M_{2k}$  of modular forms of weight  $2k$ , and find a simple basis for  $M_{2k}$  in terms of Eisenstein series, which is different from the classically known standard basis. A motivation for looking for a new basis will be explained below.

Throughout the paper, we use the following notation:

$$\begin{aligned} k & \text{ is an integer greater than or equal to } 1, \\ \Gamma & := SL_2(\mathbb{Z}) \text{ (the full modular group),} \\ M_{2k} & := \text{the } \mathbb{C}\text{-vector space of modular forms of weight } 2k \text{ on } \Gamma, \\ S_{2k} & := \text{the } \mathbb{C}\text{-vector space of cusp forms of weight } 2k \text{ on } \Gamma, \\ S_{2k}^* & := \text{Hom}_{\mathbb{C}}(S_{2k}, \mathbb{C}) \text{ (the dual space of } S_{2k}), \\ d_k & := \begin{cases} \lfloor \frac{k}{6} \rfloor - 1 & \text{if } 2k \equiv 2 \pmod{12} \\ \lfloor \frac{k}{6} \rfloor & \text{if } 2k \not\equiv 2 \pmod{12} \end{cases} \end{aligned}$$

where  $\lfloor x \rfloor$  denotes the greatest integer not exceeding  $x \in \mathbb{R}$ . We note that

$$\dim_{\mathbb{C}} S_{2k} = d_k \quad \text{and} \quad \dim_{\mathbb{C}} M_{2k} = d_k + 1.$$

Let  $B_{2n}$  is the  $2n$ th Bernoulli number and  $\sigma_{2n-1}(m)$  is the  $(2n-1)$ th divisor function. Namely,

$$\sigma_{2n-1}(m) := \sum_{0 < d \mid m} d^{2n-1} \quad (n \geq 1).$$

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Then the Eisenstein series of weight  $2n$  for  $\Gamma$  is defined by

$$G_{2n}(z) := -\frac{B_{2n}}{4n} + \sum_{m=1}^{\infty} \sigma_{2n-1}(m) e^{2\pi i m z}$$

where  $z \in \mathbb{H} := \{z \in \mathbb{C} \mid \Im(z) > 0\}$ .

The classically well-known basis for  $M_{2k}$  is the following set (Serre [8, p. 89]):

$$\{G_4^\alpha G_6^\beta \mid \alpha, \beta \in \mathbb{Z}, \alpha, \beta \geq 0, 4\alpha + 6\beta = 2k\}.$$

However, the Fourier coefficients of these forms are not so simple when we write down the coefficients as sums of products of divisor functions. This will motivate us to look for a new simpler basis for  $M_{2k}$ , consisting of modular forms whose Fourier coefficients are convolution sums of two divisor functions. Our result is formulated in the following theorem:

**Theorem 1.1.** (1) *If  $2k \equiv 0 \pmod{4}$  then*

$$\{G_{2k}\} \cup \{G_{4i} G_{2k-4i} \mid i = 1, 2, \dots, d_k\}$$

*form a basis for  $M_{2k}$ .*

(2) *If  $2k \equiv 2 \pmod{4}$  then*

$$\{G_{2k}\} \cup \{G_{4i+2} G_{2k-4i-2} \mid i = 1, 2, \dots, d_k\}$$

*form a basis for  $M_{2k}$ .*

Note that the  $n$ th Fourier coefficients of  $G_{4i} G_{2k-4i}$  is

$$\sum_{l=0}^n \sigma_{4i-1}(l) \sigma_{2k-4i-1}(n-l)$$

where we set  $\sigma_{2n-1}(0) := -B_{2n}/(4n)$  by convention.

We will also find a new basis for the space of cusp forms on  $\Gamma$  in the following theorem:

**Theorem 1.2.** (1) *If  $2k \equiv 0 \pmod{4}$  then*

$$\{G_{4i} G_{2k-4i} + \frac{B_{4i}}{4i} \frac{B_{2k-4i}}{2k-4i} \frac{k}{B_{2k}} G_{2k} \mid i = 1, 2, \dots, d_k\}$$

*form a basis for  $S_{2k}$ .*

(2) *If  $2k \equiv 2 \pmod{4}$  then*

$$\{G_{4i+2} G_{2k-4i-2} + \frac{B_{4i+2}}{4i+2} \frac{B_{2k-4i-2}}{2k-4i-2} \frac{k}{B_{2k}} G_{2k} \mid i = 1, 2, \dots, d_k\}$$

*form a basis for  $S_{2k}$ .*

We note that, for  $\Gamma = \Gamma_0(2)$ , similar but slightly different formulas were given in [4, Theorem 1.6].

**Example 1.3.** *For  $M_{36}$ , we have a basis*

$$\{G_{36}, G_4 G_{32}, G_8 G_{28}, G_{12} G_{24}\},$$

*and for  $S_{36}$ ,*

$$\begin{aligned} \{ & G_4 G_{32} - \frac{1479565184909325423}{286310154497221833818240} G_{36}, G_8 G_{28} - \frac{651138973032093}{122102860006168135010720} G_{36}, \\ & G_{12} G_{24} - \frac{114819293577343}{1149451061437375891652640} G_{36} \} \end{aligned}$$

is a basis.

## 2. PRELIMINARIES

Let  $f$  be an element of  $S_{2k}$ . We write  $f$  as a Fourier series

$$f(z) = \sum_{l=1}^{\infty} a_l e^{2\pi i l z}.$$

Let  $L(f, s)$  be the L-series of  $f$ . Namely  $L(f, s)$  is the analytic continuation of

$$\sum_{l=1}^{\infty} \frac{a_l}{l^s} \quad (\Re(s) \gg 0).$$

Then  $n$ th period of  $f$ ,  $r_n(f)$ , is defined by

$$r_n(f) := \int_0^{i\infty} f(z) z^n dz = \frac{n!}{(-2\pi i)^{n+1}} L(f, n+1) \quad (n = 0, 1, \dots, w).$$

Each period  $r_n$  can be regarded as a linear map from  $S_{2k}$  to  $\mathbb{C}$ , that is,

$$r_n \in S_{2k}^* = \text{Hom}_{\mathbb{C}}(S_{2k}, \mathbb{C}).$$

Here we recall the result of Eichler [2], Shimura [9] and Manin [6]:

**Theorem 2.1** (Eichler-Shimura-Manin). *The maps*

$$\begin{aligned} r^+ : S_{2k} &\rightarrow \mathbb{C}^k \\ f &\mapsto (r_0(f), r_2(f), \dots, r_{2k-2}(f)) \end{aligned}$$

and

$$\begin{aligned} r^- : S_{2k} &\rightarrow \mathbb{C}^{k-1} \\ f &\mapsto (r_1(f), r_3(f), \dots, r_{2k-3}(f)) \end{aligned}$$

are both injective.

In other words,

(1) the even periods

$$r_0, r_2, \dots, r_{2k-2}$$

span the vector space  $S_{2k}^*$ ;

(2) the odd periods

$$r_1, r_3, \dots, r_{2k-3}$$

also span  $S_{2k}^*$ .

However, these periods are not linearly independent. A natural question was raised in [3]: which periods form a basis for  $S_{2k}^*$ ? A satisfactory answer was obtained in the same paper [3].

To state the result in [3] we need the following notation and convention:

**Definition 2.1.** For an integer  $i$  such that  $1 \leq i \leq d_k$ , let

$$4i \pm 1 := \begin{cases} 4i + 1 & \text{if } 2k \equiv 2 \pmod{4} \\ 4i - 1 & \text{if } 2k \equiv 0 \pmod{4}. \end{cases}$$

Now we can state our result in [3]:

**Theorem 2.2** ([3]).

$$\{r_{4i\pm 1} \mid i = 1, 2, \dots, d_k\}$$

form a basis for  $S_{2k}^*$ .

Next we will display a basis for  $S_{2k}$ . For  $f, g \in S_{2k}$ , let  $(f, g)$  denote the Petersson scalar product. Then there is a cusp form  $R_n$ , which is characterized by the formula:

$$r_n(f) = (R_n, f) \quad \text{for any } f \in S_{2k}.$$

Passing to the dual space, we obtain a basis for  $S_{2k}$ .

**Theorem 2.3** ([3]).

$$\{R_{4i\pm 1} \mid i = 1, 2, \dots, d_k\}$$

form a basis for  $S_{2k}$ .

This theorem will be needed to prove Theorem 1.1. Finally some remark on the Petersson scalar product might be in order.

**Remark 2.1.** Let  $f$  and  $g$  be modular forms in  $M_{2k}$  with at least one of them a cusp form. Then the Petersson scalar product  $(f, g)$  is defined by

$$(f, g) = \int_{\Gamma/\mathbb{H}} f(z) \overline{g(z)} y^{2k-2} dx dy$$

where  $z = x + iy$ . We note that the Petersson scalar product of an Eisenstein series and a cusp form is always zero (refer to [1, p. 183]).

However, there is a natural extension of the Petersson scalar product from the space of cusp forms to the space of all modular forms (Zagier [10, pp. 434–435]). This extended scalar product is always non-degenerate, and furthermore, it is positive definite if and only if  $2k \equiv 2 \pmod{4}$ .

Petersson scalar products considered in this article are those of extended one in the above sense which are always non-degenerate.

### 3. PROOF OF THEOREMS 1.1 AND 1.2

In this section, we will give proofs of Theorems 1.1 and 1.2. We need the following lemma:

**Lemma 3.1.** *Let  $V$  be a  $\mathbb{C}$ -vector space of dimension  $n$  and*

$$B : V \times V \rightarrow \mathbb{C}$$

*be a non-degenerate bilinear form. Let*

$$\{u_i \in V \mid i = 1, \dots, n\} \quad \text{and} \quad \{v_i \in V \mid i = 1, \dots, n\}$$

*be two sets of vectors in  $V$ . Then the determinant*

$$|B(u_i, v_j)|_{i,j=1,2,\dots,n} \neq 0$$

*if and only if both  $\{u_i \in V \mid i = 1, \dots, n\}$  and  $\{v_i \in V \mid i = 1, \dots, n\}$  are sets of linearly independent vectors.*

The proof of this lemma is quite standard and we omit it.

*Proof of Theorem 1.1.* First we assume that  $2k \equiv 0 \pmod{4}$ . We consider two sets of modular forms :

$$\{G_{2k}\} \cup \{G_{4i}G_{2k-4i} \mid i = 1, 2, \dots, d_k\} \quad \text{and} \quad \{G_{2k}\} \cup \{R_{4i-1} \mid i = 1, 2, \dots, d_k\}.$$

We would like to verify that  $G_{2k}, G_{4i}G_{2k-4i}$  ( $i = 1, 2, \dots, d_k$ ) are linearly independent. By virtue of Lemma 3.1 it is sufficient to show that the determinant

$$(3.1) \quad \begin{vmatrix} (G_{2k}, G_{2k}) & (R_{4-1}, G_{2k}) & \cdots & (R_{4d_k-1}, G_{2k}) \\ (G_{2k}, G_4G_{2k-4}) & (R_{4-1}, G_4G_{2k-4}) & \cdots & (R_{4d_k-1}, G_4G_{2k-4}) \\ \cdots & \cdots & \cdots & \cdots \\ (G_{2k}, G_{4d_k}G_{2k-4d_k}) & (R_{4-1}, G_{4d_k}G_{2k-4d_k}) & \cdots & (R_{4d_k-1}, G_{4d_k}G_{2k-4d_k}) \end{vmatrix} \neq 0.$$

Since  $(G_{2k}, G_{2k}) \neq 0$  and  $(R_{4i-1}, G_{2k}) = 0$  as mentioned in Remark 2.1, (3.1) is equivalent to

$$(3.2) \quad \begin{vmatrix} (R_{4-1}, G_4G_{2k-4}) & (R_{8-1}, G_4G_{2k-4}) & \cdots & (R_{4d_k-1}, G_4G_{2k-4}) \\ (R_{4-1}, G_8G_{2k-8}) & (R_{8-1}, G_8G_{2k-8}) & \cdots & (R_{4d_k-1}, G_8G_{2k-8}) \\ \cdots & \cdots & \cdots & \cdots \\ (R_{4-1}, G_{4d_k}G_{2k-4d_k}) & (R_{8-1}, G_{4d_k}G_{2k-4d_k}) & \cdots & (R_{4d_k-1}, G_{4d_k}G_{2k-4d_k}) \end{vmatrix} \neq 0.$$

Now let  $\{f_i \mid i = 1, 2, \dots, d_k\}$  be a basis for  $S_{2k}$  such that each  $f_i$  is a normalized Hecke eigenform. Then, since  $\{R_{4i-1} \mid i = 1, 2, \dots, d_k\}$  is also a basis for  $S_{2k}$  by Theorem 2.3, we know that (3.2) is equivalent to

$$(3.3) \quad \begin{vmatrix} (f_1, G_4G_{2k-4}) & (f_2, G_4G_{2k-4}) & \cdots & (f_{d_k}, G_4G_{2k-4}) \\ (f_1, G_8G_{2k-8}) & (f_2, G_8G_{2k-8}) & \cdots & (f_{d_k}, G_8G_{2k-8}) \\ \cdots & \cdots & \cdots & \cdots \\ (f_1, G_{4d_k}G_{2k-4d_k}) & (f_2, G_{4d_k}G_{2k-4d_k}) & \cdots & (f_{d_k}, G_{4d_k}G_{2k-4d_k}) \end{vmatrix} \neq 0.$$

To show (3.3), we use the following Rankin's identity ([7], also refer to Kohnen-Zagier [5] noting that their notation of  $r_n(f)$  differs from ours by a factor  $i^{n+1}$ ): for a normalized eigenform  $f$  in  $S_{2k}$ ,

$$(3.4) \quad (f, G_{2n}G_{2k-2n}) = \frac{1}{(2i)^{2k-1}} r_{2k-2}(f) r_{2n-1}(f)$$

where  $n = 2, 3, \dots, k-2$ . From this identity we know that (3.3) is equivalent to

$$(3.5) \quad \frac{r_{2k-2}(f_1) r_{2k-2}(f_2) \cdots r_{2k-2}(f_{d_k})}{(2i)^{(2k-1)d_k}} \begin{vmatrix} r_{4-1}(f_1) & r_{4-1}(f_2) & \cdots & r_{4-1}(f_{d_k}) \\ r_{8-1}(f_1) & r_{8-1}(f_2) & \cdots & r_{8-1}(f_{d_k}) \\ \cdots & \cdots & \cdots & \cdots \\ r_{4d_k-1}(f_1) & r_{4d_k-1}(f_2) & \cdots & r_{4d_k-1}(f_{d_k}) \end{vmatrix} \neq 0.$$

Finally, (3.5) is equivalent to

$$(3.6) \quad \begin{vmatrix} (R_{4-1}, f_1) & (R_{4-1}, f_2) & \cdots & (R_{4-1}, f_{d_k}) \\ (R_{8-1}, f_1) & (R_{8-1}, f_2) & \cdots & (R_{8-1}, f_{d_k}) \\ \cdots & \cdots & \cdots & \cdots \\ (R_{d_k-1}, f_1) & (R_{d_k-1}, f_2) & \cdots & (R_{d_k-1}, f_{d_k}) \end{vmatrix} \neq 0.$$

Now (3.6) holds, since both  $\{f_i \mid i = 1, 2, \dots, d_k\}$  and  $\{R_{4i-1} \mid i = 1, 2, \dots, d_k\}$  are bases for  $S_{2k}$ . This implies the assertion (1) of Theorem 1.1.

Next we assume that  $2k \equiv 2 \pmod{4}$ . The argument similar to the above proves the assertion (2) of Theorem 1.1. This completes the proof.  $\square$

*Proof of Theorem 1.2.* In Theorem 1.1 we proved that

$$\{G_{2k}\} \cup \{G_{4i}G_{2k-4i} \mid i = 1, \dots, d_k\}$$

is a basis for  $M_{2k}$  and, in particular, the members are linearly independent. Hence  $\{G_{2k}\} \cup \{G_{4i}G_{2k-4i} + \frac{B_{4i}}{4i} \frac{B_{2k-4i}}{2k-4i} \frac{k}{B_{2k}} G_{2k} \mid i = 1, \dots, d_k\}$  are linearly independent. This implies  $\{G_{4i}G_{2k-4i} + \frac{B_{4i}}{4i} \frac{B_{2k-4i}}{2k-4i} \frac{k}{B_{2k}} G_{2k} \mid i = 1, \dots, d_k\}$  are again linearly independent. Moreover, since  $G_{4i}G_{2k-4i} + \frac{B_{4i}}{4i} \frac{B_{2k-4i}}{2k-4i} \frac{k}{B_{2k}} G_{2k} \in S_{2k}$  ( $i = 1, \dots, d_k$ ), these form a basis for  $S_{2k}$ . This completes the proof.  $\square$

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